Fairness and efficiency for probabilistic allocations with endowments

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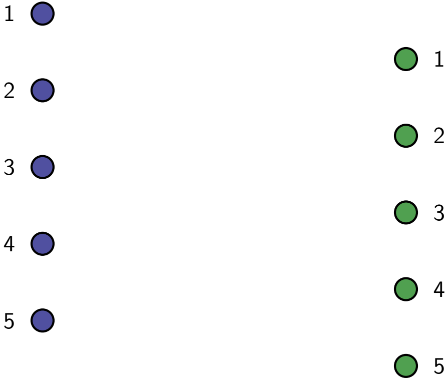
Discrete allocation

Agents: I
Resources: S (finite, indivisible).



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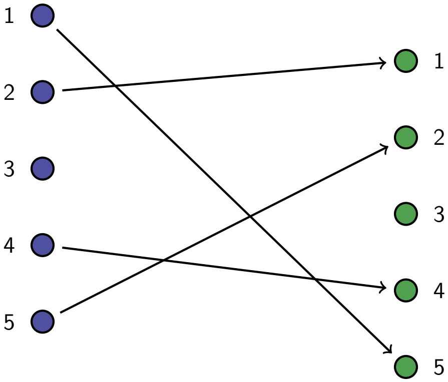

Motivation

School choice:

- Property rights are captured by priorities.
- As property rights, priorities are equivocal; not transparent.

- Endowments are explicit property rights.
- For ex., guarantee a:
  1. chance at a good school;
  2. neighborhood school;
  3. slot for a sibling.

Motivation

Arguably also why Boston abandoned the initial school choice program.

Motivation

I focus on school choice.

Another application: Time banks.

Discrete allocation

Three general objectives:

- Efficiency
- Fairness
- Respect property rights

Important challenge: how to talk about fairness for unequal agents.

School choice

Standard model:

- Property rights: school priorities
- Fairness among unequal agents: lack of justified envy.
- Efficiency and fairness are incompatible.

This paper:

- Property rights: endowments
- New notion of fairness among unequally endowed agents
- Efficiency and fairness (and respect for property rights) are compatible.
- We can handle quotas and additional constraints.

Motivation

> *". . . school choice has not delivered on a central promise: to give every student a real chance to attend a good school. Fourteen years into the system, black and Hispanic students are just as isolated in segregated high schools as they are in elementary schools — a situation that school choice was supposed to ease."*

"The Broken Promises of Choice in New York City Schools" *New York Times*, May 5th, 2017.

Eric Nadelstern at Columbia University, deputy school chancellor when school choice system was implemented, proposed to have a lottery decide the allocations. ("Confronting Segregation in New York City Schools", *New York Times*, May 15th, 2017.)

Related Literature

- Mkts. & fairness: Varian (1974), Hylland-Zeckhauser (1979), Budish (2011).
- Justified envy w/endowments: Yilmaz (2010)
- Exog. and endog. budgets: Mas-Colell (1992), McLennan (2017) and Le (2017)
- Endowments in school choice: Hamada, Hsu, Kurata, Suzuki, Ueda, and Yokoo (2017)

More references in the paper...

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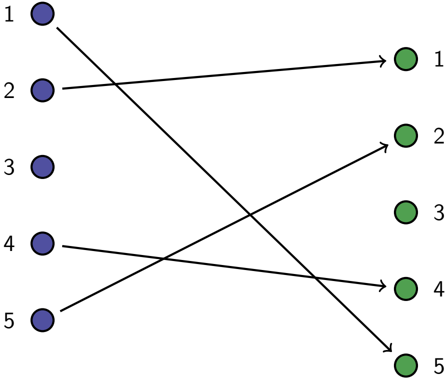

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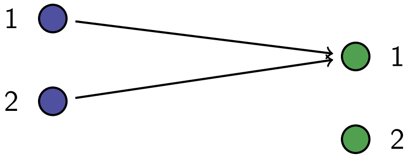

1 and 2 want school s_1.
Fairness: toss a coin.

Model

- Agents: $I = \{1, \dots, N\}$.
- Schools: $S = \{s_1, \dots, s_L\}$.
- A *lottery* is an element of

$$\Delta_- = \{x \in \mathbf{R}^L_+ : \sum_{j=1}^{L} x_j \leq 1\}$$

- $u^i : \Delta_- \to \mathbf{R}$ (cont. & mon.)

Allocations

An *allocation* is $x = (x^i)_{i=1}^N$, with $x^i \in \Delta_-^L$, s.t

$$\sum_{i \in I} x_s^i = q_s$$

Allocations

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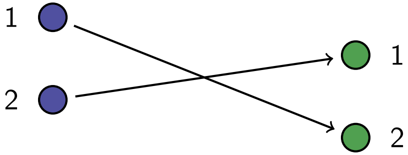
$$\sum_{i \in I} x_s^i = 1$$

Envy

i *envies* j at x if $u^i(x^j) > u^i(x^i)$

$x^i = (1/N, \ldots, 1/N) \Longrightarrow$ no envy

What if?



1 and 2 would like to *trade* probability shares.

Efficiency

An allocation x is *Pareto optimal* (PO) if there is no allocation y s.t

$$u^i(y^i) \geq u^i(x^i) \text{ for all } i \text{ and } u^j(y^j) > u^j(x^j)$$

for some j.

Fairness and efficiency

Theorem (Hylland and Zeckhauser (1979))

There is a PO and envy-free allocation. It is a market equilibrium allocation.

An *HZ-equilibrium* is a pair (x, p), with $x \in \Delta_-^N$ and $p = (p_s)_{s \in S} \geq 0$ s.t.

1. $\sum_{i=1}^N x^i = (1, \ldots, 1)$
2. x^i solves

$$\text{Max } \{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq 1\}$$

Condition (1): supply = demand.
Condition (2): x^i is i's demand at prices p and income = 1.

Observe:

- Income is independent of prices
- No endowments.

Suppose that each u^i is linear (expected utility).

Theorem (Hylland and Zeckhauser (1979))

There is a HZ equilibrium allocation. It is envy-free and PO.

Fairness among unequals

No envy makes sense for "equals."

What if agents are unequal?

In an economy with endowments, agents start from different positions.

Fairness among unequals

- Each i has an *endowment* $\omega^i \in \Delta$.
- ω^i is an initial lottery.
- Suppose that $\sum_i \omega^i = (1, \ldots, 1)$.

For example, suppose schools are allocated via a lottery. Admission probabilities reflect: neighborhood school (walk-zone priority), sibling priority, or test scores.

Walrasian equilibrium

A *Walrasian equilibrium* is a pair (x, p) with $x \in \Delta_-^N$, $p \geq 0$ s.t

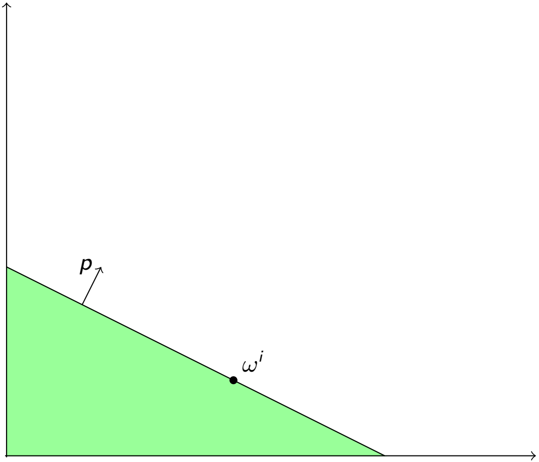
1. $\sum_{i=1}^N x^i = \sum_{i=1}^N \omega^i$; and
2. x^i solves

$$\text{Max } \{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq p \cdot \omega^i\}$$

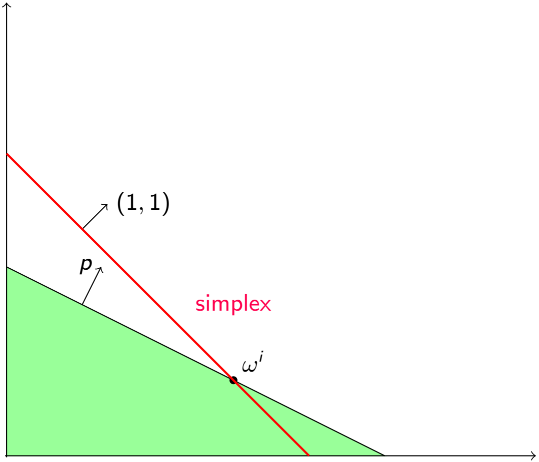
Proposition (Hylland and Zeckhauser (1979))

There are economies in which all agents' utility functions are expected utility, that posses no Walrasian equilibria.

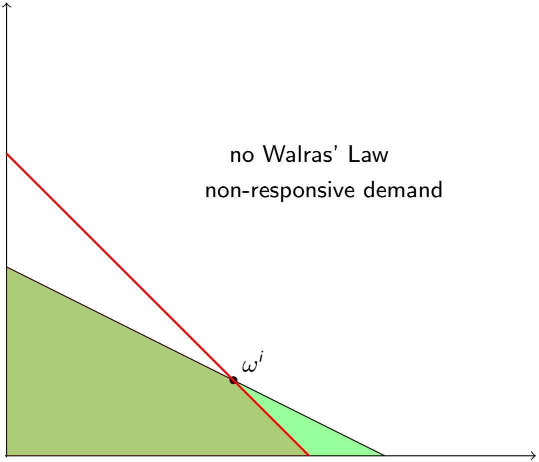
Budget set



Budget set



Budget set



HZ Example

3 agents; exp. utility

	u^1	u^2	u^3
s_A	10	10	1
s_B	1	1	10

Endowments: $\omega^i = (1/3, 2/3)$.

HZ Example

3 agents; exp. utility

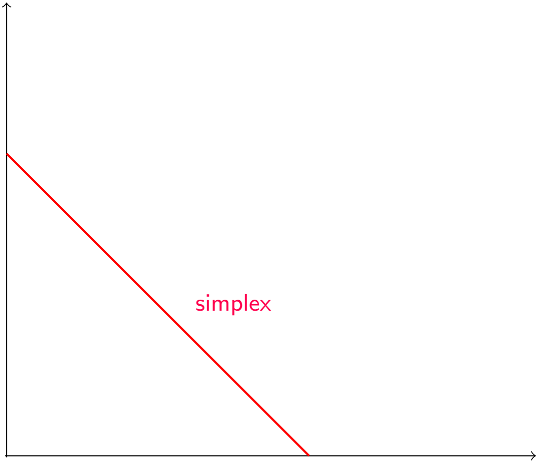
	u^1	u^2	u^3
s_A	10	10	1
s_B	1	1	10

Endowments: $\omega^i = (1/3, 2/3)$.

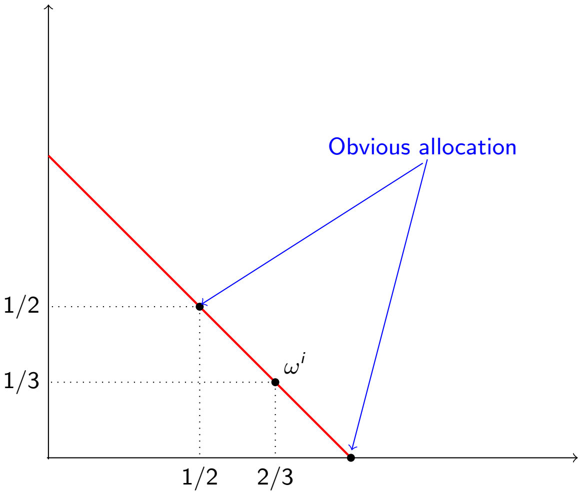
Obvious allocation:

$$x^1 = x^2 = (1/2, 1/2)$$
$$x^3 = (0, 1)$$

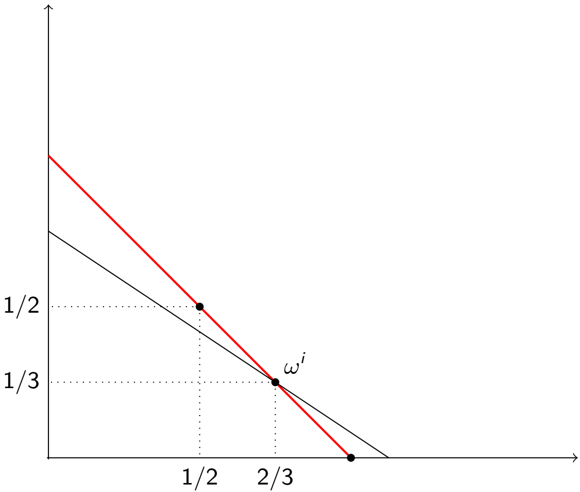
HZ Example



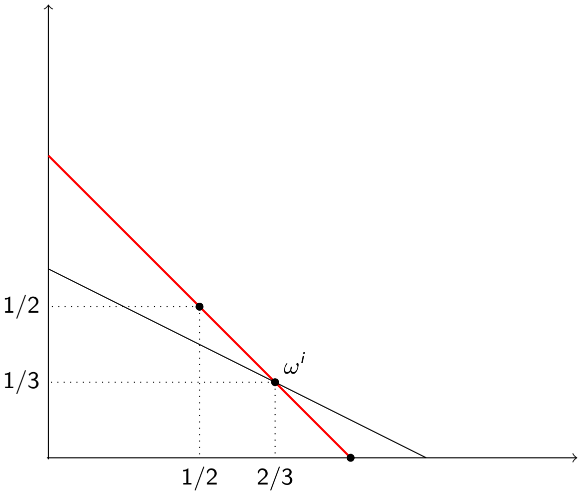
HZ Example



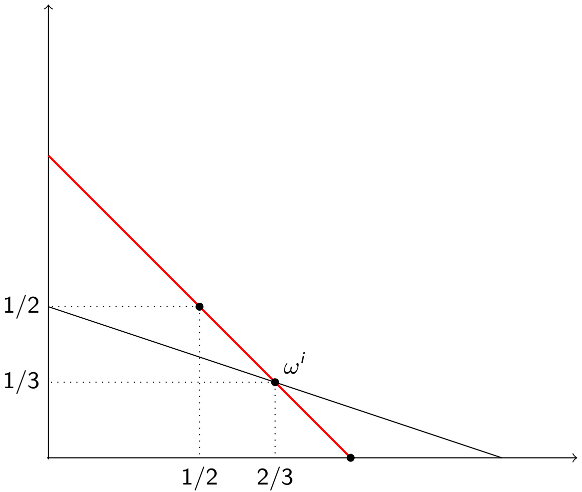
HZ Example



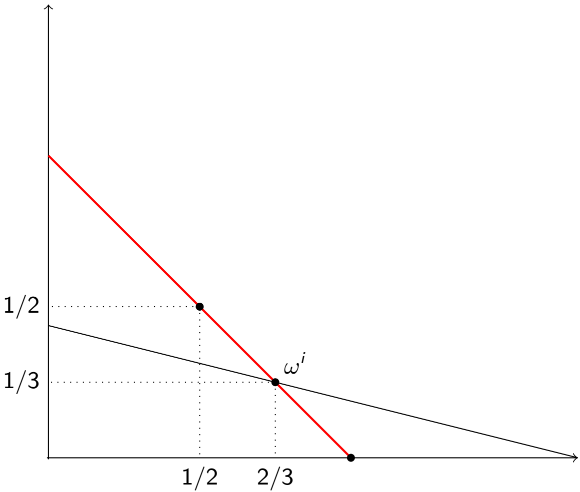
HZ Example



HZ Example



HZ Example



Moreover, . . .

- the first welfare theorem fails.
- There are Pareto ranked Walrasian equilibria.

Our results

Preliminary defns

A function $u : \Delta^n_- \to \mathbf{R}$ is

- *concave* if $\forall x, z \in \Delta_-$, and $\forall \lambda \in (0,1)$, $\lambda u(z) + (1-\lambda)u(x) \leq u(\lambda z + (1-\lambda)x)$;
- *quasi-concave* if, $\forall x \in \Delta_-$,

$$\{z \in \Delta_- : u(z) \geq u(x)\}$$

  is a convex set.
- *semi-strictly quasi-concave* if $\forall x, z \in \Delta_-$,

$$u(z) < u(x) \text{ and } \lambda \in (0,1) \implies u(z) < u(\lambda z + (1-\lambda)x)$$

- *expected utility* if it is linear.

Preliminary defns

Let x be an allocation.

- x is *weak Pareto optimal* (wPO) if $\nexists$ an allocation y s.t $u^i(y^i) > u^i(x^i)$ for all i
- *ε-weak Pareto optimal* (ε-PO), for $\varepsilon > 0$, if $\nexists$ an allocation y s.t $u^i(y^i) > u^i(x^i) + \varepsilon$ for all i.

Let x be an allocation.

- x is *acceptable* to i if $u^i(x^i) \geq u^i(\omega^i)$.
- x is *individually rational* (IR) if it is acceptable to all agents.

Justified envy

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if j's endowment is “good enough.”

Justified envy

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if j regards x^i as *unacceptable.*

Justified envy

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if $u^j(\omega^j) > u^j(x^i)$

Justified envy: defn

i has *justified envy* towards j at allocation x if

$$u^i(x^j) > u^i(x^i) \text{ and } u^j(x^i) \geq u^j(\omega^j).$$

Fairness

Let x be an allocation.

x has *no justified envy* (NJE) if no agent has justified envy towards any other agent at x.

Fairness

Observe: NJE and IR imply *equal treatment of equals*.

Fairness

Let x be an allocation.

x has *no justified envy* (NJE) if no agent has justified envy towards any other agent at x.

Justified envy

- i has *strong justified envy* (SJE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j)$.
- For $\varepsilon > 0$, i has *ε-justified envy* (ε-JE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j) - \varepsilon$.

Justified envy

no ε-justified envy $\Longrightarrow$ no justified envy $\Longrightarrow$ no strong just. envy

Fairness, property rights and efficiency

Theorem

Suppose all u^i are concave, and let $\varepsilon > 0$.

1. $\exists$ *an allocation that is* ε*-IR,* ε*-PO and has no* ε*-justified envy;*
2. $\exists$ *an allocation that is IR, wPO and has no strong justified envy.*

Idea

Consider problem

$$\text{Max} \sum_i \lambda_i u^i(x_i)$$

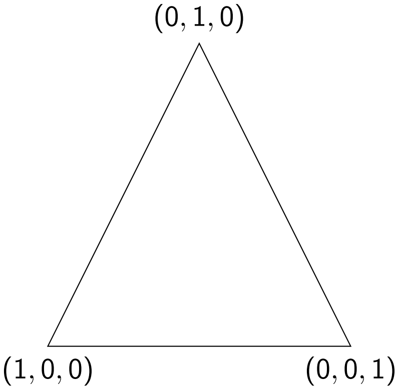
s.t. x is an allocation.

Obtain a NJE allocation from this problem

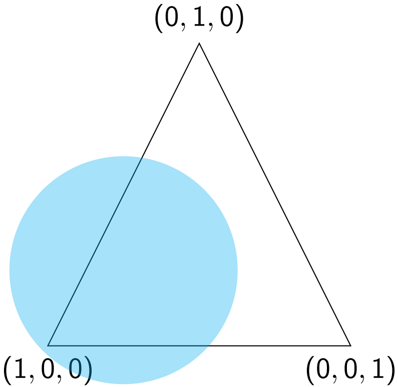
by choosing right welfare weights, $(\lambda_i) \in \Delta^N$.

(Actual proof uses an approximation to this problem, hence the ε).

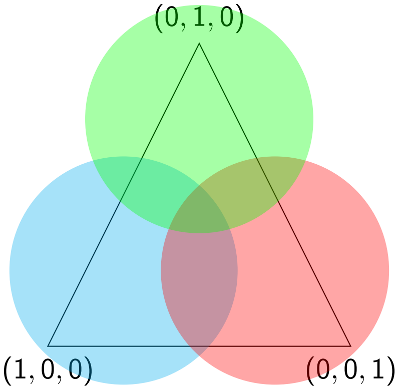
KKM Lemma



KKM Lemma



KKM Lemma



Extension 1

Constraints in assignment.

- Distributional constraints.
- Geographical constraints.
- etc.

Extension 2

i has *ε-justified envy by exchange* towards j at allocation x if there exists a sequence $(i_k)_{k=1}^{K}$ with

- $i_1 = i$ and $i_2 = j$;
- i_k envies i_{k+1} $1 \leq k \leq K - 1$
- and $u^{i_K}(x^{i_1}) > u^{i_K}(\omega^{i_K}) - \varepsilon$

Theorem

Suppose that all agents' utility functions are concave. Then, for any $\varepsilon > 0$ there exists an allocation that is ε-individually rational, ε-Pareto optimal, and has no ε-justified envy by exchange.

Proposal No. 2: Walrasian equilibrium.

Let $\alpha \in [0,1]$

An *α-slack Walrasian equilibrium* is (x, p) with $x \in \Delta_{-}^{N}$, $p \geq 0$ s.t

1. $\sum_{i=1}^{N} x^i = \sum_{i=1}^{N} \omega^i$; and
2. x^i solves

$$\text{Max } \{u^i(z^i) : z^i \in \Delta_{-} \text{ and } p \cdot z^i \leq \alpha + (1-\alpha) p \cdot \omega^i\}$$

Constraints

$$x^i_{i\leftrightarrow j} = x^j,\ x^j_{i\leftrightarrow j} = x^i, \text{ and } x^k_{i\leftrightarrow j} = x^k \text{ for all } k \in I\backslash\{i,j\}.$$

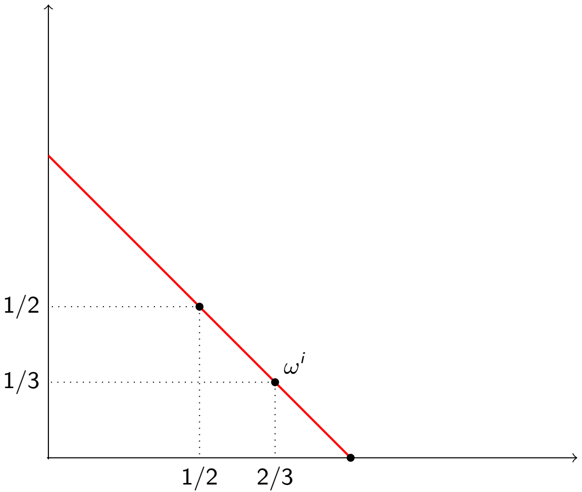
A constraint structure $\mathcal{H}$ is *anonymous* if for all $x \in \mathcal{A}^{\mathcal{H}}$ and all distinct $i, j \in I$, $x_{i\leftrightarrow j} \in \mathcal{A}^{\mathcal{H}}$.

Suppose all u^i are quasi-concave.

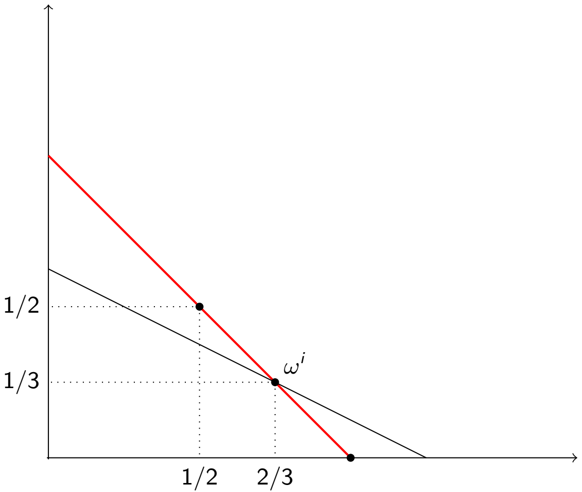
Theorem

$\forall\ \alpha \in (0,1]$, $\exists$ an α-slack Walrasian equilibrium (x,p).
Moreover, if all u^i are semi-strictly quasi-concave, then x is PO.

HZ Example



HZ Example



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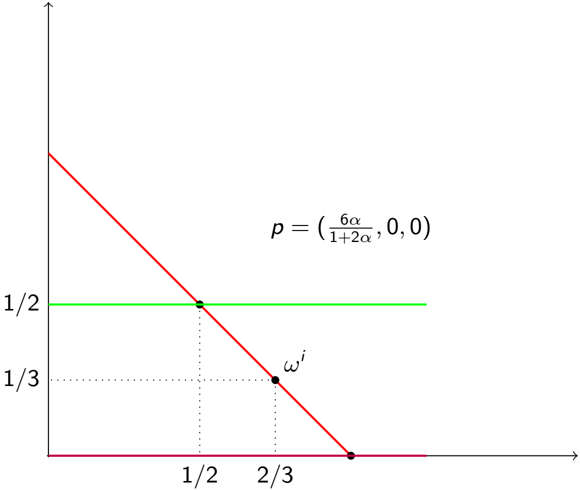


Having $\alpha > 0$ gives an "extra role" for prices, allowing agents 1 and 2 to spend above the income they get from $p \cdot \omega^i$.

That doesn't mean that ω^i don't matter ...

Suppose all u^i are semi-strictly quasi-concave.

Theorem

$\forall \varepsilon > 0 \; \exists \; \alpha \in (0, 1]$ and an α-slack Walrasian equilibrium (x, p), s.t x is PO and

$$\max\{u^i(y) : y \in \Delta_- \text{ and } p \cdot y \leq p \cdot \omega^i\} - u^i(x) < \varepsilon.$$

In particular, x is ε-individually rational.

Fairness

Let (x, p) be an α-slack Walrasian equilibrium.

If i envies j at x, then $p \cdot \omega^j > p \cdot \omega^i$.

"Society" values j's endowment more than i's.

Suppose all u^i are concave and C^1. Let (x, p) be an α-slack Walrasian equilibrium. Denote by $S = \{i : u^i(x^i) = \max\{u^i(z^i) : z^i \in \Delta_-\}\}$ the set of *satiated* consumers, and by $U = [N] \setminus S$.
Suppose that $\sum_{i \in U} x^i \gg 0$.

Theorem

If i envies j at x then $p \cdot \omega^j > p \cdot \omega^i$,

Suppose all u^i are concave and C^1. Let (x, p) be an α-slack Walrasian equilibrium. Denote by $S = \{i : u^i(x^i) = \max\{u^i(z^i) : z^i \in \Delta_-\}\}$ the set of *satiated* consumers, and by $U = [N] \setminus S$.
Suppose that $\sum_{i \in U} x^i \gg 0$.

Theorem

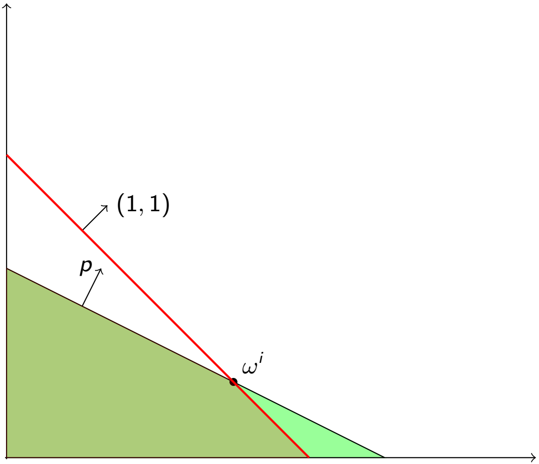
If i envies j at x then $p \cdot \omega^j > p \cdot \omega^i$, and $\exists$ welfare weights $\theta \in \mathbf{R}^U_{++}$ s.t

$$v(t) = \sup\{\sum_{i \in U} \theta^i u^i(\tilde{x}^i) : (\tilde{x}^i) \in \Delta^U_- \text{ and } \sum_{i \in U} \tilde{x}^i \leq \bar{\omega} + t(\omega^i - \omega^j) - \sum_{i \in S} x^i\},$$

then $(x^i)_{i \in U}$ solves the problem for $v(0)$, and $v(t) < v(0)$ for all t small enough.

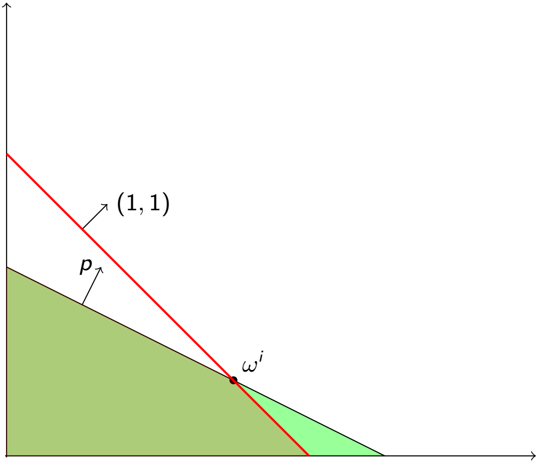
Idea

Classical result relies on Walras Law: $p \cdot z(p) = 0$ for all p. Walras Law does not hold in our model because...



Idea

Demand is not responsive to price once boundary is reached.



Idea

Budget constraint:

$$p \cdot x^i \leq \alpha + (1 - \alpha) p \cdot \omega^i$$

Idea

Budget constraint:

$$p \cdot (x^i - \omega^i) \leq \alpha(1 - p \cdot \omega^i).$$

This allows prices to matter: large prices imply that the value of excess demand is < 0.

Idea

Consider $\phi : [0, \bar{p}]^L \to [0, \bar{p}]^L$ defined by

$$\phi_l(p) = \{\min\{\max\{0, \zeta_l + p_l\}, \bar{p}\} : \zeta \in z(p)\}.$$

where $\bar{p}$ is a large price.

Lemma

ϕ is upper hemi-continuous, convex- and compact- valued.

(In paper deal with a different ϕ, which ensures PO.)

Idea

By Kakutani, $\exists$ p^* and $\zeta \in z(p^*)$ s.t

$$p_l^* = \min\{\max\{0, \zeta_l + p_l^*\}, \bar{p}\}.$$

Lemma

$p^* \cdot \zeta \geq 0$.

This is sort of a "weak Walras law."

Pf: $\zeta_l < 0 \Longrightarrow p_l^* = 0$

Idea

Lemma

$p_l^* < \bar{p}$ *for all* $l \in [L]$

Pf: Suppose $p_l^* = \bar{p}$. $\bar{p}$ is large $\implies 1 - p \cdot \omega^i < 0$; so $p \cdot (x^i - \omega^i) < 0$.
By adding up we get that

$$p \cdot \zeta \leq \alpha(N - p \cdot \bar{\omega}) < 0,$$

in contradiction to prev. lemma.

Idea

Now think about:

$$p_l^* = \min\{\max\{0, \zeta_l + p_l^*\}, \bar{p}\}.$$

when $p_l^* < \bar{p}$.

we have

$$p_l^* = \max\{0, \zeta_l + p_l^*\}.$$

Idea

$$p_l^* = \max\{0, \zeta_l + p_l^*\}.$$

For all l, $\zeta_l = 0$, or $\zeta_l < 0$ and $p_l^* = 0$.

Latter case is not possible.

Pareto ranked equilibria

Two agents and two schools.

i	$u^i_{s_1}$	$u^i_{s_2}$
1	1	1
2	1	100

Endowments $\omega^i = (1/2, 1/2)$.

Consider the allocations
$x = ((1,0),(0,1))$ and
$y = ((1/2,1/2),(1/2,1/2))$.
Note that x Pareto dominates y.

Pareto ranked equilibria

α	allocation	p	$\alpha + (1-\alpha)p \cdot \omega^i$
0	x	$(1,1)$	1
$1/2$	x	$(1,1)$	1
0	y	$(0,1)$	$1/2$
$1/2$	y	$(0,2)$	1

First welfare theorem

Theorem

Any Walrasian equilibrium with slack is weakly Pareto optimal, and any Walrasian equilibrium with slack and the cheapest-bundle property is Pareto optimal.

Related Literature

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- Endowments in school choice: Hamada, Hsu, Kurata, Suzuki, Ueda, and Yokoo (2017)

Conclusions

- New criteria for discrete (probabilistic) allocation.
- Model with *explicit* property rights.
- Fairness among unequals.
- Market equilibrium
- Fairness, efficiency and property rights.